THE IDEAL OF p-COMPACT OPERATORS: A TENSOR PRODUCT APPROACH

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Abstract. We study the space of p-compact operators $\mathcal{K}_p$, using the theory of tensor norms and operator ideals. We prove that $\mathcal{K}_p$ is associated to $/d_p$, the left injective associate of the Chevet-Saphar tensor norm d_p (which is equal to $g'_{p'}$). This allows us to relate the theory of p-summing operators with that of p-compact operators. With the results known for the former class and appropriate hypothesis on E and F we prove that $\mathcal{K}_p(E;F)$ is equal to $\mathcal{K}_q(E;F)$ for a wide range of values of p and q, and show that our results are sharp. We also exhibit several structural properties of $\mathcal{K}_p$. For instance, we obtain that $\mathcal{K}_p$ is regular, surjective, totally accessible and characterize its maximal hull $\mathcal{K}_p^{max}$ as the dual ideal of the p-summing operators, Π_p^{dual}. Furthermore, we prove that $\mathcal{K}_p$ coincides isometrically with $\mathcal{QN}_p^{dual}$, the dual ideal of the quasi p-nuclear operators.

Introduction

In 1956, Grothendieck published his famous Resume [8] in which he set out a complete theory of tensor products of Banach spaces. In the years following, the parallel theory of operator ideals was initiated by Pietsch [11]. Researchers in the field have generally preferred the language of operator ideals to the more abstruse language of tensor products, and so the former theory has received more attention in the succeeding decades. However, the monograph of Defant and Floret [3], in which the two fields are described in tandem, has initiated a period in which authors use indistinctly both languages.

In the recent years, Sinha and Karn [15] introduced the notion of (relatively) p-compact sets. The definition is inspired in Grothendieck's result which characterize relatively compact sets as those contained in the convex hull of a norm null sequence of vectors of the space. In a similar form, p-compact sets are determined by norm p-summable sequences. Related to this concept, the ideal of p-compact operators $\mathcal{K}_p$, and different approximation properties naturally appear (see definitions below). Since relatively p-compact sets are, in particular, relatively compact, p-compact operators are compact. These concepts were first studied in [15] and thereafter in several other articles, see for instance [1, 2, 4, 5, 6, 16]. However,

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we believe that the goodness of the space of p-compact operators inherits from the general theory of operator ideals and tensor products has not yet been fully exploited.

The main purpose of this article is to show that the principal properties of the class of p-compact operators can be easily obtained if we study this operator ideal with the theory of tensor products and tensor norms. This insight allows us to give new results, to recover many already known facts, and also to improve some of them.

The paper is organized as follows. In Section 1 we fix some notation and list the classical operator ideals, with their associated tensor norms, which we use thereafter. Section 2 is devoted to general results on p-compact sets and p-compact operators. We define a measure m_p, to study the size of a p-compact set K in a Banach space E and show that this measure is independent of whether K is considered as a subset of E or as a subset of E'', the bidual of E. This allows us to show that $\mathcal{K}_p$ is regular. In addition, we prove that $\mathcal{K}_p$ coincides isometrically with $\mathcal{QN}_p^{dual}$, the dual ideal of the quasi p-nuclear operators. We also show that any p-compact operator factors via a p-compact operator and two other compact operators.

In Section 3 we use the Chevet-Saphar tensor norm d_p to find the appropriate tensor norm associated to the ideal of p-compact operators. We show that $\mathcal{K}_p$ is associated to the left injective associate of d_p, denoted by $/d_p$, which is equal to $g'_{p'}$. We use this to link the theory of p-summing operators with that of p-compact operators. With the results known for the former class and natural hypothesis on E and F we show that $\mathcal{K}_p(E;F)$ and $\mathcal{K}_q(E;F)$ coincide for a wide range of values of p and q. We also use the limit orders of the ideals of p-summing operators to show that our results are sharp. Furthermore, we prove that $\mathcal{K}_p$ is surjective, totally accessible and characterize its maximal hull $\mathcal{K}_p^{max}$ as the dual ideal of the p-summing operators, Π_p^{dual}.

For the sake of completeness, we list as an Appendix the limit orders of the ideal p-compact operators obtained by a simple transcription of those given in [11] for p-summing operators.

When the final version of this manuscript was being written, we got to know a preprint on the same subject authored by Albrecht Pietsch [12]. The main results in both articles coincide. However, the material in each paper was obtained independently. While A. Pietsch based his work using the classical theory of operator ideals following his monograph [11], we preferred the language of tensor products developed in the book by A. Defant and K. Floret [3].

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1. Notation and Preliminaries

Along this paper E and F denote Banach spaces, E' and B_E denote respectively the topological dual and the closed unit ball of E. A sequence $(x_n)_n$ in E is said to be p-summable if $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ and $(x_n)_n$ is said to be weakly p-summable if $\sum_{n=1}^{\infty} |x'(x_n)|^p < \infty$ for all $x' \in E'$. We denote, respectively, $\ell_p(E)$ and $\ell_p^w(E)$ the spaces of all p-summable and weakly p-summable sequences of E, $1 \leq p < \infty$. Both spaces are Banach spaces, the first one endowed with the norm $\|(x_n)_n\|_p = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p}$ and the second if the norm $\|(x_n)_n\|_p^w = \sup\limits_{x' \in B_{E'}} \{(\sum_{n=1}^{\infty} |x'(x_n)|^p)^{1/p}\}$ is considered. For $p = \infty$, we have the spaces $c_0(E)$ and $c_0^w(E)$ formed, respectively, by all null and weakly null sequences of E, endowed with the natural norms. The p-convex hull of a sequence $(x_n)_n$ in $\ell_p(E)$ is defined as $p\text{-co}\{x_n\} = \{\sum_{n=1}^{\infty} \alpha_n x_n \colon (\alpha_n)_n \in B_{\ell_{p'}}\}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ ($\ell_{p'} = c_0$ if $p = 1$).

Following [15], we say that a subset $K \subset E$ is relatively p-compact, $1 \leq p \leq \infty$, if there exists a sequence $(x_n)_n \subset \ell_p(E)$ so that $K \subset p\text{-co}\{x_n\}$.

The space of linear bounded operators from E to F is denoted by $\mathcal{L}(E;F)$ and its subspace of finite rank operators by $\mathcal{F}(E;F)$. Often the finite rank operator $x \mapsto \sum_{j=1}^{n} x_j'(x) y_j$ is associated with the element $\sum_{j=1}^{n} x_j' \otimes y_j$ in $E' \otimes F$. In many cases, the completion of $E' \otimes F$ with a reasonable tensor norms produces a subspace of $\mathcal{L}(E;F)$. For instance the injective tensor product $E' \hat{\otimes}_\varepsilon F$ can be viewed as the approximable operators from E to F. The Chevet-Saphar tensor norm g_p defined on $E' \otimes F$ by $g_p(u) = \inf\{\|(x_n')_n\|_p \|(y_n)_n\|_{p'}^w \colon u = \sum_{j=1}^{n} x_j' \otimes y_j\}$, gives the ideal of p-nuclear operators $\mathcal{N}_p(E;F)$, $1 \leq p \leq \infty$. If we denote $x' \underline{\otimes} y$ the 1-rank operator $x \mapsto x'(x)y$, we have that

$$\mathcal{N}_p(E;F) = \{T = \sum_{n=1}^{\infty} x_n' \underline{\otimes} y_n \colon (x_n')_n \in \ell_p(E') \text{ and } (y_n)_n \in \ell_{p'}^w(F)\},$$

is a Banach operator ideal endowed with the norm

$$v_p(T) = \inf\{\|(x_n')_n\|_p \|(y_n)_n\|_{p'}^w \colon T = \sum_{n=1}^{\infty} x_n' \underline{\otimes} y_n\}.$$

It is known that the space of p-nuclear operators is a quotient of $E' \hat{\otimes}_{g_p} F$ and the equality $\mathcal{N}_p(E;F) = E' \hat{\otimes}_{g_p} F$ holds if either E' or F has the approximation property, see [14, Chapter 6]. The definition of g_p is not symmetric, its transpose $d_p = g_p^t$ is associated with the operator ideal

$$\mathcal{N}^p(E;F) = \{T = \sum_{n=1}^{\infty} x_n' \underline{\otimes} y_n \colon (x_n')_n \in \ell_{p'}^w(E') \text{ and } (y_n)_n \in \ell_p(F)\},$$

equipped with the norm

$$v^p(T) = \inf\{\|(x_n')_n\|_{\ell_{p'}^w(E')} \|(y_n)_n\|_{\ell_p(F)} \colon T = \sum_{n=1}^{\infty} x_n' \underline{\otimes} y_n\}.$$

Here, we have that $\mathcal{N}^p(E;F) = E'\hat{\otimes}_{d_p}F$ if either E' or F has the approximation property. Also, note that when $p = 1$, we obtain $\mathcal{N}_1 = \mathcal{N}^1 = \mathcal{N}$, the ideal of nuclear operators and $d_1 = g_1 = \pi$, the projective tensor norm.

In this paper, we are focused on the study of p-compact operators, introduced by Sinha and Karn [15] as those which map the closed unit ball into p-compact sets. The space of p-compact operators is denoted by $\mathcal{K}_p(E;F)$, $1 \leq p \leq \infty$ which is an operator Banach ideal endowed with the norm

$$\kappa_p(T) = \inf\{\|(x_n)_n\|_p \colon T(B_E) \subset p\text{-co}\{x_n\}\}.$$

We want to understand this operator ideal in terms of a tensor product and a reasonable tensor norm. In order to do so we also make use of the ideal of the quasi p-nuclear operators introduced and studied by Persson and Pietsch [13]. The space of quasi p-nuclear operators from E to F is denoted by $\mathcal{QN}_p(E;F)$. This ideal is associated by duality with the ideal of p-compact operators [6].

Recall that an operator T is quasi p-nuclear if and only if there exists a sequence $(x'_n)_n \subset \ell_p(E')$, such that

$$\|Tx\| \leq \Big(\sum_n |x'_n(x)|^p\Big)^{\frac{1}{p}},$$

for all $x \in E$ and the quasi p-nuclear norm of T is given by $v_p^Q(T) = \inf\{\|(x'_n)_n\|_p\}$, where the infimum is taken over all the sequences $(x'_n)_n \in \ell_p(E')$ satisfying the inequality above. It is known that $\mathcal{QN}_p = \mathcal{N}_p^{inj}$, where $\mathcal{N}_p^{inj}$ denotes the injective hull of $\mathcal{N}_p$.

For general background on tensor products and tensor norms we refer the reader to the monographs by Defant and Floret [3] and by Ryan [14]. All the definitions and notation we use regarding tensor norms and operator ideals can be found in [3]. For further reading on operator ideals we refer the reader to Pietsch's book [11].

2. On p-compact sets and p-compact operators

Given a relatively p-compact set K in a Banach space E there exists a sequence $(x_n)_n \subset E$ so that $K \subset p\text{-co}\{x_n\}$. Such a sequence is not unique, then we may consider the following definition.

Definition 2.1. *Let E be a Banach space, $K \subset E$ a p-compact set, $1 \leq p \leq \infty$. We define*

$$m_p(K;E) = \inf\{\|(x_n)_n\|_{\ell_p(E)} \colon K \subset p\text{-co}\{x_n\}\}.$$

If $K \subset E$ is not a p-compact set, $m_p(K;E) = \infty$.

We say that $m_p(K;E)$ measures the size of K as a p-compact set of E.

There are some properties which derive directly from the definition of m_p. For instance, since p-co$\{x_n\}$ is absolutely convex, $m_p(K;E) = m_p(\overline{\text{co}}\{K\};E)$. Also, by Hölder's inequality, we have that $\|x\| \leq \|(x_n)_n\|_{\ell_p(E)}$ and in consequence, $\|x\| \leq m_p(K)$, for all $x \in K$. Moreover, as compact sets can be considered p-compact sets for $p = \infty$ we have that any p-compact set is q-compact and $\sup_{x\in K} \|x\| = m_\infty(K;E) \leq m_q(K;E) \leq m_p(K;E)$, for $1 \leq p \leq q \leq \infty$.

Some other properties are less obvious. Suppose that E is a subspace of another Banach space F. It is clear that if $K \subset E$ is p-compact in E then K is p-compact in F and $m_p(K;F) \leq m_p(K;E)$. As we see in Section 3, the definition of m_p depends on the space E. In other words, K may be p-compact in F but not in E. We show this in Corollary 3.5.

For the particular case when $F = E''$, the bidual of E, Delgado, Piñeiro and Serrano [6, Corollary 3.6] show that a set $K \subset E$ is p-compact if only if K is p-compact in E'' with $m_p(K;E'') \leq m_p(K;E)$. We want to prove that, in fact, the equality $m_p(K;E'') = m_p(K;E)$ holds. In order to do so we propose to inspect various results concerning operators and their adjoint and show that the transpose operator is not only continuous but also an isometry.

Recall that when E' has the approximation property, any operator $T \in \mathcal{L}(E;F)$ with nuclear adjoint T', is nuclear and both nuclear norms coincide, $\nu(T) = \nu(T')$, see for instance [14, Proposition 4.10]. The analogous result for p-nuclear operators is due to Reinov [10, Theorem 1] and states that when E' has the approximation property and T' belongs to $\mathcal{N}_p(F';E')$, then $T \in \mathcal{N}^p(E;F)$. However, the relationship between $\nu^p(T)$ and $\nu_p(T')$ is omitted. It is clear that whenever T is in $\mathcal{N}^p(E;F)$ its adjoint is p-nuclear and, in that case, $\nu_p(T') \leq \nu^p(T)$. The Proposition 2.3 below shows that the isometric result is also valid for p-nuclear operators. Before showing this, we need the following result.

Proposition 2.2. *Let E and F be Banach spaces, E' with the approximation property, and let $T \in \mathcal{L}(E;F)$. If $J_F T \in \mathcal{N}^p(E;F'')$ then $T \in \mathcal{N}^p(E;F)$ and $\nu^p(J_F T) = \nu^p(T)$.*

Proof. We only need to show the equality of the norms, the first part of the assertion corresponds with the first statement of [10, Theorem 1]. Note that since E' has the approximation property, then $\mathcal{N}^p(E;F) = E'\hat{\otimes}_{d_p}F$ and $\mathcal{N}^p(E;F'') = E'\hat{\otimes}_{d_p}F''$.By the embedding lemma $E'\hat{\otimes}_{d_p}F$ is a subspace of $E'\hat{\otimes}_{d_p}F''$ via $Id_{E'} \otimes J_F$. Therefore,

$$\nu^p(J_F T) = \nu^p(T),$$

and the proof is complete. $\square$

Proposition 2.3. *Let E be a Banach space such that E' has the approximation property and let $1 \leq p < \infty$. If $T \in \mathcal{L}(E;F)$ has p-nuclear adjoint, then $T \in \mathcal{N}^p(E;F)$ and $\nu^p(T) = \nu_p(T')$.*

Proof. The first part of the assertion is a direct consequence of [10, Theorem 1]. We only give the proof which shows the isometric result. Take T as in the statement. Since T' belongs to $\mathcal{N}_p(F';E')$, there exist sequences $(y''_n)_n \in \ell_p(F'')$ and $(x'_n)_n \in \ell^w_{p'}(E')$ such that $T' = \sum_{n=1}^{\infty} y''_n \underline{\otimes} x'_n$. Then, $J_F T = T'' J_E = \sum_{n=1}^{\infty} x'_n \underline{\otimes} y''_n$, which implies that $J_F T \in \mathcal{N}^p(E;F'')$. It is clear that $\nu_p(T') \geq \nu^p(J_F T)$. By Proposition 2.2 we have that $T \in \mathcal{N}^p(E;F)$ and $\nu^p(J_F T) = \nu^p(T)$. The reverse inequality always holds, whence the result follows. □

Now we are ready to prove that the $\mathfrak{m}_p$-measure of a p-compact set $K \subset E$ does not change if K is considered as a subset of E''.

Theorem 2.4. *Let E be a Banach space and $K \subset E$. Then K is p-compact in E if and only if K is p-compact in E'' and $\mathfrak{m}_p(K;E) = \mathfrak{m}_p(K;E'')$.*

Proof. We only need to show the inequality $\mathfrak{m}_p(K;E) \leq \mathfrak{m}_p(K;E'')$ since the claim K is p-compact in E if and only if K is p-compact in E'' is proved in [6, Corollary 3.6]. Also, in this case, the inequality $\mathfrak{m}_p(K;E'') \leq \mathfrak{m}_p(K;E)$ is obvious.

Suppose that $K \subset E$ is p-compact and define the operator $\Psi\colon \ell_1(K) \to E$ such that for $\alpha = (\alpha_x)_{x\in K}$,

$$\Psi(\alpha) = \sum_{x\in K} \alpha_x x.$$

Note that $K \subset \Psi(B_{\ell_1(K)}) \subset \overline{\text{co}}(K)$ thus, Ψ and $J_E\Psi$ are p-compact operators. Also, $\mathfrak{m}_p(K;E) = \kappa_p(\Psi)$ and $\mathfrak{m}_p(K;E'') = \kappa_p(J_E\Psi)$. By [6, Proposition 3.1], $\Psi' J'_E$ belongs to $\mathcal{QN}_p(E''';\ell_\infty(K))$ and $\nu_p^Q(\Psi' J'_E) \leq \kappa_p(J_E\Psi)$. Therefore Ψ' belongs to $\mathcal{QN}_p(E';\ell_\infty(K))$ and $\nu_p^Q(\Psi') \leq \nu_p^Q(\Psi' J'_E)$.

Since $\ell_\infty(K)$ is injective, $\Psi' \in \mathcal{N}_p(E';\ell_\infty(K))$ and $\nu_p(\Psi') = \nu_p^Q(\Psi')$, see [13, Satz 38]. Now, an application of Proposition 2.3, gives us that Ψ is an operator in $\mathcal{N}^p(\ell_1(K);E)$ and $\nu^p(\Psi) = \nu_p(\Psi')$. In particular, $\Psi \in \mathcal{K}_p(\ell_1(K);E)$ and $\kappa_p(\Psi) \leq \nu^p(\Psi)$.

Thus, we have

$$\mathfrak{m}_p(K;E) = \kappa_p(\Psi) \leq \nu^p(\Psi) = \nu_p(\Psi') = \nu_p^Q(\Psi') \leq \nu_p^Q(\Psi' J'_E) \leq \kappa_p(J_E\Psi) = \mathfrak{m}_p(K;E''),$$

and the proof is complete. □

Theorem 2.5. *The operator ideal $\mathcal{K}_p$ is regular.*

Proof. Let E and F be Banach spaces and $T\colon E \to F$ be an operator such that $J_F T$ is p-compact. Therefore, by the theorem above, $\mathfrak{m}_p(J_F T(B_E);F'') = \mathfrak{m}_p(T(B_E);F)$ and T is p-compact. Whence, the result follows. □

Also we obtain the isometric version of [6, Corollary 3.6] which is stated as follows.

Corollary 2.6. *Let E and F be Banach spaces. Then $T \in \mathcal{K}_p(E;F)$ if and only if $T'' \in \mathcal{K}_p(E'';F'')$ and $\kappa_p(T) = \kappa_p(T'')$.*

Proof. The statement $T \in \mathcal{K}_p(E;F)$ if and only if $T'' \in \mathcal{K}_p(E'';F'')$ is part of [6, Corollary 3.6]. Let T be a p-compact operator. In particular, $T(B_E)$ is relatively compact and

$$J_F T(B_E) \subset T''(B_{E''}) \subset \overline{J_F T(B_E)}^{w^*} = \overline{J_F T(B_E)}.$$

Applying twice Theorem 2.4 we get

$$m_p(T(B_E);F) = m_p(T(B_E);F'') \leq m_p(T''(B_{E''});F'') \leq m_p(\overline{J_F T(B_E)};F'') = m_p(T(B_E);F).$$

Since $\kappa_p(T) = m_p(T(B_E);F)$ and $\kappa_p(T'') = m_p(T''(B_{E''});F'')$, the isometry is proved. $\square$

Now, we describe the duality between p-compact and quasi p-nuclear operators. On the one hand, we have that an operator T is quasi p-nuclear if and only if T' is p-compact and $\kappa_p(T') = \nu_p^Q(T)$ [6, Corollary 3.4]. On the other hand, T is p-compact if and only if its adjoint T' is quasi p-nuclear, in this case $\nu_p^Q(T') \leq \kappa_p(T)$ [6, Proposition 3.8]. We improve this last result by showing that the transposition is an isometry.

Corollary 2.7. *Let E and F be Banach spaces. Then $T \in \mathcal{K}_p(E;F)$ if and only if $T' \in \mathcal{QN}_p(F';E')$ and $\kappa_p(T) = \nu_p^Q(T')$.*

Proof. The inequality $\nu_p^Q(T') \leq \kappa_p(T)$ and the equality $\kappa_p(T'') = \nu_p^Q(T')$ always hold. A direct application of Corollary 2.6 completes the proof. $\square$

With Corollary 2.7 and the results mentioned above we can state the following identities.

Theorem 2.8. $\mathcal{K}_p^{dual} \stackrel{1}{=} \mathcal{QN}_p$ *and* $\mathcal{QN}_p^{dual} \stackrel{1}{=} \mathcal{K}_p$.

We finish this section with a factorization result of p-compact operators, which improves [15, Theorem 3.2] and [2, Theorem 3.1]. The characterization given below should be compared with [7, Proposition 5.23].

Proposition 2.9. *Let E and F be Banach spaces. Then an operator $T \in \mathcal{L}(E;F)$ is p-compact if and only if T admits a factorization via a p-compact operator T_0 and a two compact operators R and S such that $T = ST_0R$.*

Moreover, $\kappa_p(T) = \inf\{\|S\|\kappa_p(T_0)\|R\|\}$ where the infimum is taken over all the factorizations as above.

Proof. Suppose that T belongs to $\mathcal{K}_p(E;F)$. Then, given $\varepsilon > 0$, there exists $y = (y_n)_n \in \ell_p(F)$ such that $T(B_E) \subset p\text{-co}\{y_n\}$, with $\|(y_n)_n\|_p \leq \kappa_p(T)(1+\varepsilon)$. We may choose $\beta = (\beta_n)_n \in B_{c_0}$ such that $(\frac{y_n}{\beta_n})_n \in \ell_p(F)$ and $\|(\frac{y_n}{\beta_n})_n\|_p \leq \|(y_n)_n\|_p(1+\varepsilon)$. Now, with $z =$

$(z_n)_n = (\frac{y_n}{\beta_n})_n$, $T(B_E) \subset \{\sum_{n=1}^{\infty} \alpha_n z_n \colon (\alpha_n)_n \in L\}$ where L is a compact set in $B_{\ell_{p'}}$. By the factorization given in [15, Theorem 3.2], we have the commutative diagram

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[dr, "R"'] & F & \ell_{p'} \arrow[l, "\theta_z"'] \arrow[dl, "\pi"] \\ & \ell_{p'}/\ker\theta_z \arrow[u, "\tilde{\theta}_z"] & \end{tikzcd}$$

where π is the projection mapping and, θ_z and R are given by $\theta_z((\alpha_n)_n) = \sum_{n=1}^{\infty} \alpha_n z_n$ and $R(x) = [(\alpha_n)_n]$ where $(\alpha_n)_n \in L$ is a sequence satisfying that $T(x) = \sum_{n=1}^{\infty} \alpha_n z_n$. Since $R(B_E) = \pi(L)$, we see that R is compact and $T = \tilde{\theta}_z R$.

Note also that $\tilde{\theta}_z$ is p-compact. Since $\|R\| \leq 1$, then

$$\kappa_p(T) \leq \kappa_p(\tilde{\theta}_z) \leq \|(z_n)_n\|_p \leq \kappa_p(T)(1+\varepsilon)^2.$$

Now, using [2, Theorem 3.1] we factorize $\tilde{\theta}_z$ via a p-compact operator T_0 and a compact operator S, as follows:

$$\begin{tikzcd} \ell_{p'}/\ker\theta_z \arrow[rr, "\tilde{\theta}_z"] \arrow[dr, "T_0"'] & & F \\ & \ell_1/M \arrow[ur, "S"'] & \end{tikzcd}$$

where M is a closed subspace of ℓ_1. A close inspection to the proof given in [2] allows us to chose the a factorization such that $\kappa_p(\tilde{\theta}_z) \leq \|S\|\kappa_p(T_0) \leq (1+\varepsilon)\kappa_p(\tilde{\theta}_z)$, (just consider a sequence $(\beta_n)_n$ similar to that used above). Whence, the factorization is obtained together with the desired equality $\kappa_p(T) = \inf\{\|S\|\kappa_p(T_0)\|R\|\}$.

The reverse claim is obvious. □

Note that if both E' and F have the approximation property then T belongs to $\mathcal{K}_p(E;F)$ if and only if T belongs to $\mathcal{K}_p^{min}(E;F)$. Moreover, $\kappa_p(T) = \kappa_p^{min}(T)$. We show in the next section that the same result holds if only one of the spaces (E' or F) has the approximation property.

3. Tensor norms

Our purpose in this section is to draw together the theory of operator ideals and tensor products for the class of p-compact operators. To start with our aim we use the Chevet-Saphar tensor norm to find the appropriate tensor norm associated to the ideal of p-compact operators. The tensor norm obtained is $g'_{p'}$ which allows us to connect the theory of p-summing operators with that of p-compact operators. With the results known for the former class, with additional hypothesis on E and F we show that $\mathcal{K}_p(E;F)$ and $\mathcal{K}_q(E;F)$ coincide

for a wide range of p and q. We also use the limit orders of the ideal of p-summing operators [11] to show that the values considered for p and q cannot be improved. Some other properties describing the structure of the ideal of p-compact operators are given.

Recall that $d_p(u) = \inf\{\|(x_n)_n\|_{p'}^w \|(y_n)_n\|_p\}$ where the infimum is taken over all the possible representations of $u = \sum_{j=1}^n x_j \otimes y_j$. We denote by $/d_p$ the left injective tensor norm associated to d_p. Note that $/d_p = g'_{p'}$ [14, Theorem 7.20] and therefore $/d_p = (g^*_{p'})^t$.

Proposition 3.1. *The ideal $(\mathcal{K}_p, \kappa_p)$ is surjective.*

Proof. Let $Q\colon G \overset{1}{\twoheadrightarrow} E$ be a quotient map, if TQ is p-compact, then $TQ(B_G) = T(B_E)$ is a p-compact set. Thus, T is p-compact and

$$\kappa_p(TQ) = m_p(TQ(B_G)) = m_p(T(B_E)) = \kappa_p(T).$$

$\square$

In order to characterize the tensor norm associated to $(\mathcal{K}_p, \kappa_p)$ we need the following simple lemma. We sketch its proof for completeness. This result should be compared with [3, Theorem 20.11].

Lemma 3.2. *Let $(\mathcal{A}, \|.\|_{\mathcal{A}})$ be an operator ideal and let α be its associated tensor norm.*

(a) *If $\mathcal{A}$ is surjective then, α is left injective.*
(b) *If $\mathcal{A}$ is injective then, α is right injective.*

Proof. Suppose $\mathcal{A}$ is surjective. Using a 'left version' of [3, Proposition 20.3 (1)], we only need to see that α is left injective on FIN, the class of all finite dimensional spaces.

Fix $N, M, W \in FIN$ such that $i\colon M \overset{1}{\hookrightarrow} W$, then we have the commutative diagram

$$\begin{tikzcd}
M \otimes_\alpha N \arrow[r, "i \otimes id_N"] \arrow[d, equal] & W \otimes_\alpha N \arrow[d, equal] \\
\mathcal{A}(M'; N) \arrow[r, "\phi"'] & \mathcal{A}(W'; N)
\end{tikzcd}$$

where ϕ is given by $T \mapsto Ti'$. As i is an isometry, i' is a metric surjection. Now, since $\mathcal{A}$ is surjective ϕ is an isometry, which proves (a).

The proof of (b) follows easily with a similar reasoning. $\square$

From [6, Proposition 3.11] we have $\mathcal{N}^p(\ell_1^n; N) \overset{1}{=} \mathcal{K}_p(\ell_1^n; N)$, for every n and every finite dimensional space N. Since $\mathcal{N}^p$ is associated to the tensor norm d_p, we have the following result.

Theorem 3.3. *The operator ideal $(\mathcal{K}_p, \kappa_p)$ is associated to the tensor norm $/d_p$, for every $1 \leq p < \infty$.*

Proof. Denote by α the tensor norm associated to $\mathcal{K}_p$. By Proposition 3.1 and the above lemma, α is left injective. Note that for every n and every finite dimensional space N we have the isometric identities

$$\ell_\infty^n \otimes_{d_p} N = \mathcal{N}^p(\ell_1^n; N) = \mathcal{K}_p(\ell_1^n; N) = \ell_\infty^n \otimes_\alpha N.$$

Now, applying a 'left version' of [3, Proposition 20.9], we conclude that $\alpha = /d_p$. □

Proposition 3.4. *The operator ideal $(\mathcal{K}_p, \kappa_p)$ is not injective, for any $1 \leq p < \infty$.*

Proof. Suppose that $\mathcal{K}_p$ is injective. By Theorem 3.3 and Lemma 3.2 we see that $/d_p$ the associated tensor norm to $\mathcal{K}_p$, is right injective. Thus, its transpose $g_{p'}^*$ is left injective. Now, by [3, Theorem 20.11], Π_p is surjective which is a contradiction. Note that, by Grothendieck's theorem [3, Theorem 23.10], $id\colon \ell_2 \to \ell_2$ belongs to Π_p^{sur} and obviously is not p-summing. □

As a consequence we show that the $\mathfrak{m}_p$-measure of a set, depends on the space which contains the set.

Corollary 3.5. *Given $1 \leq p < \infty$, there exist a Banach space G, a subspace $F \subset G$ and a set $K \subset F$ such that K is p-compact in G but K fails to be p-compact in F.*

Proof. Since $(\mathcal{K}_p, \kappa_p)$ in not injective, there exist Banach spaces E, F and G, $F \overset{I_{F,G}}{\hookrightarrow} G$ and an operator $T \in \mathcal{L}(E; F)$ such that $I_{F,G}T$ is p-compact but T is not. Taking $K = T(B_E)$, we see that $\mathfrak{m}_p(K; G) < \infty$ while $\mathfrak{m}_p(K; F) = \infty$. □

Now we characterize $\mathcal{K}_p^{max}$, the maximal hull of the operator ideal $\mathcal{K}_p$ in terms of the ideal of p-summing operators Π_p.

Corollary 3.6. *The operator ideal $\mathcal{K}_p^{max}$ coincides isometrically with Π_p^{dual}.*

Proof. The maximal hull of $\mathcal{K}_p$ is also associated to the tensor norm $/d_p = (g_{p'}^*)^t$. Since the ideal of p-summing operators Π_p is associated to the tensor norm $g_{p'}^*$, by Corollary 3 in [3, 17.8] the result follows. □

By [3, Proposition 21.1 (3)] and the fact that $/(d_p/) = /d_p$ we see that the tensor norm $/d_p$ is totally accessible (see also [14, Corollary 7.15]). Therefore, we have the following two results. For the first one we use [3, Proposition 21.3] and for the second one we use [3, Corollary 22.2].

Remark 3.7. *The operator ideal $\mathcal{K}_p^{max} \overset{1}{=} \Pi_p^{dual}$ is totally accessible.*

Remark 3.8. *For any Banach spaces E and F, $\mathcal{K}_p^{min}(E;F) \overset{1}{=} E'\widehat{\otimes}_{/d_p}F$.*

With the help of Corollary 3.6 we obtain an easy way to compute the κ_p norm of a p-compact operator: just take the p-summing norm of its adjoint. Moreover, the same holds for the minimal norm. We also have the following isometric relations.

Proposition 3.9. *The isometric inclusions hold*

$$\mathcal{K}_p^{min} \overset{1}{\hookrightarrow} \mathcal{K}_p \overset{1}{\hookrightarrow} \mathcal{K}_p^{max} \overset{1}{=} \Pi_p^{dual}.$$

In particular, $\mathcal{K}_p^{min}$ and $\mathcal{K}_p$ are totally accessible.

Proof. Let E and F be Banach spaces. We have

$$\mathcal{K}_p^{min}(E;F) \overset{\leq 1}{\hookrightarrow} \mathcal{K}_p(E;F) \overset{\leq 1}{\hookrightarrow} \mathcal{K}_p^{max}(E;F) \overset{1}{=} \Pi_p^{dual}(E;F).$$

Now, using the previous remark and [3, Corollary 22.5], we have $\mathcal{K}_p^{min}(E;F) \overset{1}{\hookrightarrow} \mathcal{K}_p^{max}(E;F) \overset{1}{=} \Pi_p^{dual}(E;F)$, which implies that all the inclusions above are isometries. $\square$

The definition of the κ_p-approximation property was given in terms of operators in [5]: A Banach space F has the κ_p-approximation property if, for every Banach space E, $\mathcal{F}(E;F)$ is κ_p-dense in $\mathcal{K}_p(E;F)$. In other words,

$$\overline{\mathcal{F}(E;F)}^{\kappa_p} \overset{1}{=} \mathcal{K}_p(E;F).$$

On the other hand, by Remark 3.7, [3, Corollary 22.5] and the previous proposition we have

$$\mathcal{K}_p^{min}(E;F) \overset{1}{=} \overline{\mathcal{F}(E;F)}^{\mathcal{K}_p^{max}} \overset{1}{=} \overline{\mathcal{F}(E;F)}^{\kappa_p}.$$

Therefore, F has the κ_p-approximation property if and only if $\mathcal{K}_p^{min}(E;F) \overset{1}{=} \mathcal{K}_p(E;F)$, for every Banach space E.

Any Banach space with the approximation property enjoys the κ_p-approximation property. This result can be deduced from [5, Theorem 3.1]. Below, we give a short proof using the language of operator ideals. It is worthwhile mentioning that every Banach space has the κ_2-approximation property (which can be deduced from [15, Theorem 6.4]) and for each $p \neq 2$ there exists a Banach space whose dual lacks the κ_p-approximation property [5, Theorem 2.4].

Proposition 3.10. *If a Banach space has the approximation property then it has the κ_p-approximation property.*

Proof. We have shown that a Banach space F has the κ_p-approximation property if and only if $\mathcal{K}_p^{min}(E;F) \overset{1}{=} \mathcal{K}_p(E;F)$, for every Banach space E. Suppose that F has the approximation

property and let $T \in \mathcal{K}_p(E;F)$. Using [2, Theorem 3.1] we have the following factorization

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "T_0"'] & & F \\ & G \arrow[ur, "S"'] & \end{tikzcd} ,$$

where T_0 is p-compact and S is compact (therefore approximable). Now, by [3, Proposition 25.2 (1) b.], T belongs to $\mathcal{K}_p^{min}(E;F)$, which concludes the proof. $\square$

Note that, in general, the converse of Proposition 3.10, is not true. For instance, if $1 \leq p < 2$, we always may find a subspace $E \subset \ell_q$, $1 < q < 2$ without the approximation property. This subspace is reflexive and has cotype 2. Then, by the comment bellow [3, Proposition 21.7] and applying [6, Corollary 2.5] to obtain that $F = E'$ has the κ_p-approximation property and it fails to have the approximation property.

In this setting, the next theorem becomes quite natural. It states that the ideal of p-compact operators can be represented in terms of tensor products in presence of the κ_p-approximation property.

Theorem 3.11. *Let E and F be Banach spaces. Then,*

$$E'\widehat{\otimes}_{/d_p} F \overset{1}{=} \mathcal{K}_p(E;F)$$

if and only if F has the κ_p-approximation property. Also, the isometry remains valid whenever E' has the approximation property, regardless of F.

Proof. Note that $/d_p$ is totally accessible (see the comments preceding Remark 3.7). Thus, the proof of the first claim is straightforward from Remark 3.8.

For the second statement, take $T \in \mathcal{K}_p(E;F)$. By Proposition 2.9, $T = T_0 R$ where R is a compact operator and T_0 is p-compact. Now, using that E' has the approximation property, R is approximable by finite rank operators and an application of [3, Proposition 25.2 (2) b.] gives that $T \in \mathcal{K}_p^{min}(E;F)$. Again, the result follows by Remark 3.8. $\square$

The next result improves [5, Proposition 3.3].

Corollary 3.12. *Let E and F be Banach spaces such that F has the κ_p-approximation property or E' has the approximation property . Then, $\mathcal{K}_p(E;F)' \overset{1}{=} \mathcal{I}_{p'}(E';F')$, $1 \leq p \leq \infty$.*

Proof. The proof is straightforward from Theorem 3.11 and [14, Pag 174]. $\square$

Now, we compare p-compact and q-compact operators for certain classes of Banach spaces. We use some well known results stated for p-summing operators when the spaces involved are of finite cotype or $\mathcal{L}_{q,\lambda}$-spaces, for some q. Our results are stated in terms of $\mathcal{K}_p^{min}(E;F)$

but if F has the κ_p-approximation property or E' has the approximation property, by Theorem 3.11, they can be stated for $\mathcal{K}_p(E;F)$. First we need the following general result. As usual, for $s=\infty$, we consider $\mathcal{L}(X;Y)$ instead of $\Pi_s(X;Y)$ and $\overline{\mathcal{F}(Y;X)}$ instead of $\mathcal{K}_s^{min}(Y;X)$.

Theorem 3.13. *Let X and Y be Banach spaces such that for some $1 \leq r < s \leq \infty$ $\Pi_r(X';Y') = \Pi_s(X';Y')$. Then, $\mathcal{K}_s^{min}(Y;X) = \mathcal{K}_r^{min}(Y;X)$.*

Moreover, if $\pi_r(\cdot) \leq A\pi_s(\cdot)$ on $\Pi_s(X';Y')$ then $\kappa_r(\cdot) \leq A\kappa_s(\cdot)$ on $K_s^{min}(Y;X)$, $A>0$.

Proof. Suppose that $\Pi_r(X';Y') = \Pi_s(X';Y')$. Since Π_r is a maximal ideal and its associated tensor norms, $g_{r'}^*$ is totally accessible [3, Corollary 21.1.] we have, by the Embedding Theorem [3, 17.6.], $X''\widehat{\otimes}_{g_{r'}^*}Y' \overset{1}{\hookrightarrow} \Pi_r(X';Y')$. Now, using the Embedding lemma [3, 13.3.] we have the following commutative diagram

$$\begin{tikzcd}
Y'\widehat{\otimes}_{/d_s}X = X\widehat{\otimes}_{g_{s'}^*}Y' \arrow[r, hook, "1"] \arrow[d, dashed, "\leq A"] & X''\widehat{\otimes}_{g_{s'}^*}Y' \arrow[r, hook, "1"] & \Pi_s(X';Y') \arrow[d, hook, "\leq A"] \\
Y'\widehat{\otimes}_{/d_r}X = X\widehat{\otimes}_{g_{r'}^*}Y' \arrow[r, hook, "1"] & X''\widehat{\otimes}_{g_{r'}^*}Y' \arrow[r, hook, "1"] & \Pi_r(X';Y')
\end{tikzcd}$$

Therefore, $/d_s \leq /d_r \leq A\ /d_s$ on $Y'\otimes X$, which implies that $\mathcal{K}_s^{min}(Y;X) = \mathcal{K}_r^{min}(Y;X)$ and $\kappa_r(T) \leq A\kappa_s(T)$ for every $T \in K_s^{min}(Y;X)$. □

In order to compare the norm $\kappa_r(T)$ with $\|T\|$ or with $\kappa_s(T)$, we use the constants obtained in comparing summing operators, taken from [17]. Some of them involve the Grothendieck constant K_G, the constant B_r taken from Khintchine's inequality and $C_q(E)$ the q-cotype constant of E. With this notation and the theorem above we have the following results.

Corollary 3.14. *Let E and F be Banach spaces such that E is a $\mathcal{L}_{2,\lambda'}$-space and F is a $\mathcal{L}_{\infty,\lambda}$-space. Then, $\overline{\mathcal{F}(E;F)} = \mathcal{K}_1^{min}(E;F)$ and $\kappa_1(T) \leq K_G\lambda\lambda'\|T\|$ for every $T \in \overline{\mathcal{F}(E;F)}$.*

Proof. Note that E is a $\mathcal{L}_{2,\lambda'}$-space if and only if E' is a $\mathcal{L}_{2,\lambda'}$-space and F is a $\mathcal{L}_{\infty,\lambda}$-space if and only if F' is a $\mathcal{L}_{1,\lambda}$-space, see [3, 23.2 Corollary 1] and [3, 23.3]. Now, use Theorem 3.13 with [3, Theorem 23.10] or [17, Theorem 10.11]. □

Corollary 3.15. *Let E and F be Banach spaces such that F is a $\mathcal{L}_{1,\lambda}$-space. Then,*

(a) *if E' has cotype 2, $\overline{\mathcal{F}(E;F)} = \mathcal{K}_2(E;F) = \mathcal{K}_r^{min}(E;F)$, for all $2 \leq r$ and*

$$\kappa_r(T) \leq \lambda\left[c\,C_2(E')^2\,(1+\log C_2(E'))\right]^{1/r}\|T\|,$$

for all $T \in \mathcal{K}_r^{min}(E;F)$.

(b) *if E' has cotype q, $2 < q < \infty$, $\overline{\mathcal{F}(E;F)} = \mathcal{K}_r^{min}(E;F)$ for all $q < r < \infty$ and*

$$\kappa_r(T) \leq \lambda \; c \; q^{-1}(1/q - 1/r)^{-1/r'} C_q(E')\|T\|,$$

for all $T \in \mathcal{K}_r^{min}(E;F)$.

In each case, $c > 0$ is a universal constant.

Proof. Again, F is a $\mathcal{L}_{\infty,\lambda}$-space if and only if F' is a $\mathcal{L}_{1,\lambda}$-space. For the first statement, note that every space has the κ_2-approximation property, $\mathcal{K}_2^{min}(E;F) = \mathcal{K}_2(E;F)$. Now, use Theorem 3.13 with the combination of [17, Theorem 10.14] and [17, Proposition 10.16]. For the second claim, use Theorem 3.13 and [17, Theorem 21.4 (ii)]. □

Corollary 3.16. *Let E and F be Banach spaces. Then,*

(a) *if E' has cotype 2, $\mathcal{K}_r^{min}(E;F) = \mathcal{K}_2(E;F)$, for all $2 \leq r < \infty$ and*

$$\kappa_2(T) \leq B_r C_2(E')\kappa_r(T),$$

for every $T \in \mathcal{K}_r^{min}(E;F)$.

(b) *if F' has cotype 2, $\mathcal{K}_2^{min}(E;F) = \mathcal{K}_1^{min}(E;F)$, for all E and*

$$\kappa_1(T) \leq c\, C_2(F')(1 + \log C_2(F'))^{1/2}\kappa_2(T),$$

for every $T \in \mathcal{K}_2^{min}(E;F)$.
In particular, for all $1 \leq r \leq 2$, $\mathcal{K}_r^{min}(E;F) = \mathcal{K}_1^{min}(E;F)$, for all E.

(c) *if F' has cotype q, $2 < q < \infty$, $\mathcal{K}_r^{min}(E;F) = \mathcal{K}_1^{min}(E;F)$, for all $1 \leq r < q'$ and for all E, and*

$$\kappa_1(T) \leq c\, q^{-1}(1/q - 1/r')^{-1/r} C_q(F')\kappa_r(T),$$

for every $T \in \mathcal{K}_r^{min}(E;F)$.

In each statement $c > 0$ is a universal constant.

Note that if E' and F' have cotype 2, $\mathcal{K}_r^{min}(E;F) = \mathcal{K}_1^{min}(E;F)$, for all $1 \leq r < \infty$.

Proof. Use Theorem 3.13 and [17, Theorem 10.15] for part (a). For (b) use [17, Corollary 10.18 (a)]. Finally, use Theorem 3.13 with [17, Corollary 21.5 (i)] for the third claim. □

We finish this section by showing that the conditions considered on r in the corollaries above are sharp. We make use of the notion of limit order [11, Chapter 14], which has proved useful, specially to compare different operator ideals. Recall that for an operator ideal $\mathcal{A}$, the limit order $\lambda(\mathcal{A}, u, v)$ is defined to be the infimum of all $\lambda \geq 0$ such that the diagonal operator D_λ belongs to $\mathcal{A}(\ell_u;\ell_v)$, where $D_\lambda\colon (a_n) \mapsto (n^{-\lambda}a_n)$ and $1 \leq u, v \leq \infty$.

Lemma 3.17. *Let $1 \le u, v, p \le \infty$ and u', v', p' the respective conjugates. Then,*

$$\lambda(\mathcal{K}_p, u, v) = \lambda(\Pi_p, v', u').$$

Proof. Denote by $id_{u,v}$ the identity map from ℓ_u^n to ℓ_v^n, for a fixed integer n. By Corollary 3.6 we have

$$\kappa_p(id_{u,v} \colon \ell_u^n \to \ell_v^n) = \pi_p(id_{v',u'} \colon \ell_{v'}^n \to \ell_{u'}^n).$$

Then, a direct application of [11, Theorem 14.4.3] gives the result. $\square$

Now we have:

Result 3.18. *The conditions imposed on r in Corollary 3.15 and Corollary 3.16 are sharp.*

(1) Let $E = \ell_u$ and $F = \ell_1$. Note that (see Appendix (a) and (b))

$$\lambda(\mathcal{K}_r, u, 1) = \begin{cases} 1 - \frac{1}{u} & \text{if} \quad r' \le u \le \infty, \\ \frac{1}{r} & \text{if} \quad 1 \le u \le r'. \end{cases}$$

Fixed $1 \le r < 2$ choose $2 < u < r'$, then E' has cotype 2 and $\lambda(\mathcal{K}_r, u, 1) = \frac{1}{r} \neq \frac{1}{u'} = \lambda(\mathcal{K}_2, u, 1)$. Thus, $\mathcal{K}_r(\ell_u; \ell_1) \neq \mathcal{K}_2(\ell_u; \ell_1)$ and r cannot be included in Corollary 3.15 (a), whenever $1 \le r < 2$.

Now, fix $2 < q$ and let $E = \ell_{q'}$. Then E' has cotype q and given $r < q$, we see that $\lambda(\mathcal{K}_r, q', 1) = \frac{1}{r}$. On the other hand, $\lambda(\mathcal{K}_s, q', 1) = \frac{1}{q}$ for any $q < s$. This shows that $\mathcal{K}_r(\ell_{q'}; \ell_1) \neq \mathcal{K}_s(\ell_{q'}; \ell_1)$ for any $r < q < s$.
Note that we have also shown that if $r < \tilde{r} \le q$, then $\lambda(\mathcal{K}_{\tilde{r}}, q', 1) \neq \lambda(\mathcal{K}_r, q', 1)$. Therefore, the inclusions $\mathcal{K}_{\tilde{r}}(\ell_{q'}, \ell_1) \subset \mathcal{K}_r(\ell_{q'}, \ell_1)$ are always strict, for any $r < \tilde{r} \le q$.

For the case $r = q$, $2 < q < \infty$, take $E = L_{q'}[0,1] = L_{q'}$ and $F = L_1[0,1] = L_1$. Suppose that $\overline{\mathcal{F}(L_{q'}; L_1)} = \mathcal{K}_q(L_{q'}; L_1)$. By Theorem 3.11, $L_q \widehat{\otimes}_{/d_q} L_1 = L_q \widehat{\otimes}_\varepsilon L_1$ and $L_1 \widehat{\otimes}_{(/d_q)^t} L_q = L_1 \widehat{\otimes}_\varepsilon L_q$. Since $(/d_q)^t = (d_{q'})'$, and $\pi' = \varepsilon$, then $L_1 \widehat{\otimes}_{d'_{q'}} L_q = L_1 \widehat{\otimes}_{\pi'} L_q$ and we get that $(L_1 \widehat{\otimes}_{d'_{q'}} L_q)' = (L_1 \widehat{\otimes}_{\pi'} L_q)'$. Since both L_∞ and $L_{q'}$ have the metric approximation property, by [3, 17.7] and [3, 12.4], we have the isomorphism $L_\infty \widehat{\otimes}_{d_{q'}} L_{q'} = L_\infty \widehat{\otimes}_\pi L_{q'}$. Therefore $(L_\infty \widehat{\otimes}_{d_{q'}} L_{q'})' = (L_\infty \widehat{\otimes}_\pi L_{q'})'$. In other words, $\Pi_q(L_\infty, L_q) = \mathcal{L}(L_\infty, L_q)$ (see [14, Section 6.3]). This last equality contradicts [9, Theorem 7].

(2) For any $1 \le p < \infty$, there exits a compact operator in $\mathcal{L}(\ell_p; \ell_p)$ (and therefore approximable), which is not p-compact [1, Example 3.1]. Thus, $\overline{\mathcal{F}(\ell_p; \ell_p)} \neq \mathcal{K}_p(\ell_p; \ell_p)$. Fix $p \ge 2$, for all $2 \le r < \infty$, we see that $\mathcal{K}_r^{min}(\ell_p; \ell_p) = \mathcal{K}_p^{min}(\ell_p; \ell_p) = \mathcal{K}_p(\ell_p; \ell_p) = \mathcal{K}_2(\ell_p; \ell_p)$, Corollary 3.16 (a). Then, $r = \infty$ cannot be included in the first statement of this corollary.

Also, for $r < 2$, we may choose p and q such that $2 \leq p \leq r'$ and $1 \leq q \leq r$. Now, with $E = \ell_p$ and $F = \ell_q$ we compute the limit orders (see Appendix) obtaining $\lambda(\mathcal{K}_r, p, q) = \frac{1}{r}$ and $\lambda(\mathcal{K}_2, p, q) = \frac{1}{2}$, we conclude that the inclusion $\mathcal{K}_r(\ell_p; \ell_q) \subset \mathcal{K}_2(\ell_p; \ell_q)$ is strict.

(3) To see that the choice of r in Corollary 3.16 (b) is sharp, fix $r > 2$. Take p and q such that $2 \leq q < r$ and $1 \leq p \leq r'$. Let $E = \ell_p$ and $F = \ell_{q'}$, using the limit orders we obtain $\lambda(\mathcal{K}_2, p, q') = \frac{1}{2}$ and $\lambda(\mathcal{K}_r, p, q') = \frac{1}{r}$ (see Appendix (b)). Thus, $\mathcal{K}_2(\ell_p; \ell_{q'}) \neq \mathcal{K}_r(\ell_p; \ell_{q'})$.
Here, we have also shown that if $2 \leq r < \tilde{r}$, the inclusions $\mathcal{K}_{\tilde{r}}(\ell_p; \ell_{q'}) \subset \mathcal{K}_r(\ell_p; \ell_{q'})$ are strict, for suitable p and q.

(4) Now, we focus our attention on Corollary 3.16 (c). Fix $2 < q$ and let $E = \ell_1$ and $F = \ell_{q'}$. We claim that $\mathcal{K}_r(\ell_1, \ell_{q'}) \neq \mathcal{K}_1(\ell_1, \ell_{q'})$, for any $q' < r$. In fact, the result follows using the limit orders: $\lambda(\mathcal{K}_1, 1, q') = \frac{1}{q'}$ and $\lambda(\mathcal{K}_r, 1, q') = \frac{1}{r}$. This also shows that $\mathcal{K}_{\tilde{r}}(\ell_1; \ell_{q'})$ is strictly contained in $\mathcal{K}_r(\ell_1; \ell_{q'})$ for any $q' \leq r < \tilde{r}$.

Finally, we deal with the remaining case, $r = q'$. Take $E = L_1[0,1] = L_1$, $F = L_{q'}[0,1] = L_{q'}$ and suppose that $\mathcal{K}_{q'}(L_1; L_{q'}) = \mathcal{K}_1(L_1; L_{q'})$, $2 < q < \infty$. Applying Theorem 3.11 we get that $L_\infty \widehat{\otimes}_{g'_q} L_{q'} = L_\infty \widehat{\otimes}_{g'_\infty} L_{q'}$. Thus, the tensor spaces have isomorphic duals. By [3, 17.7] and [3, 13.3] we obtain the isomorphism $L_1 \widehat{\otimes}_{g_q} L_q = L_1 \widehat{\otimes}_{g_\infty} L_q$. Since $g_\infty = \backslash\varepsilon$ and $g_q = \backslash g^*_{q'}$, by [3, Corollary 1 20.6], $L_1 \widehat{\otimes}_{g^*_{q'}} L_q = L_1 \widehat{\otimes}_\varepsilon L_q$. As shown in part (1), this cannot happen.

4. Appendix

In this section we transcribe the values of $\lambda(\Pi_r, v', u')$, computed in Pietsch's monograph, to give $\lambda(\mathcal{K}_r, u, v)$. To this end, we combine Propositions 22.4.9, 22.4.12 and 22.4.13 in [11].

(a) For $1 \leq r \leq 2$,

$$\lambda(\mathcal{K}_r, u, v) = \begin{cases} \frac{1}{r} & \text{if} \quad 1 \leq v \leq r, \quad 1 \leq u \leq r', \\ 1 - \frac{1}{u} & \text{if} \quad 1 \leq v \leq r, \quad r' \leq u \leq \infty, \\ \frac{1}{v} & \text{if} \quad r \leq v \leq 2, \quad 1 \leq u \leq v', \\ 1 - \frac{1}{u} & \text{if} \quad r \leq v \leq 2, \quad v' \leq u \leq \infty, \\ \frac{1}{v} & \text{if} \quad 2 \leq v \leq \infty, \quad 1 \leq u \leq 2, \\ \frac{1}{2} - \frac{1}{u} + \frac{1}{v} & \text{if} \quad 2 \leq v \leq \infty, \quad 2 \leq u \leq \infty. \end{cases}$$

(b) For $2 < r < \infty$,

$$\lambda(\mathcal{K}_r, u, v) = \begin{cases} \frac{1}{r} & \text{if} \quad 1 \leq v \leq r, \quad 1 \leq u \leq r', \\ 1 - \frac{1}{u} & \text{if} \quad 1 \leq v \leq 2, \quad r' \leq u \leq \infty, \\ \rho & \text{if} \quad 2 \leq v \leq r, \quad r' \leq u \leq 2, \\ \frac{1}{v} & \text{if} \quad r \leq v \leq \infty, \quad 1 \leq u \leq 2, \\ \frac{1}{2} - \frac{1}{u} + \frac{1}{v} & \text{if} \quad 2 \leq v \leq \infty, \quad 2 \leq u \leq \infty, \end{cases}$$

where $\rho = \dfrac{1}{r} + \dfrac{(\frac{1}{v} - \frac{1}{r})(\frac{1}{r'} - \frac{1}{u})}{\frac{1}{2} - \frac{1}{r}}$.

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